# The integration of the sum of measurable positive functions.

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f$ and $g$ be two positive measurable functions. We will prove that the integration of $f + g$ is the same as the sum of the integration of $f$ and $g$, where the integration of positive measurable functions is defined as in Rudin book.

**Remark:** This result is not trivial at all. In Rudin's book, such heavy machinery as the Lebesgue Montone Convergence Theorem is used to prove this result.

Here we will give a relative rudimentary, but not simple, proof from the definitions.

In fact, during the class, using the defition of integrations of positive measurable functions, which says that

$$\int_X f \,\mathrm{d}\mu = \left\{ \int_X s \,\mathrm{d}\mu \colon s \text{ is simple}, 0 \le s \le f \right\}$$

and

$$\int_X g \,\mathrm{d}\mu = \left\{ \int_X s \,\mathrm{d}\mu \colon s \text{ is simple}, 0 \le s \le g \right\},$$

considering all the simple functions $f_n$ and $g_n$, satisfying $0 \le f_n \le g$ and $0 \le g_n \le g$, and noting that it then follows $0 \le f_n + g_n \le f + g$, we already proved the following proposition.

**Prop 1.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f$ and $g$ be two positive measurable functions. Then

$$\int_X f + g \,\mathrm{d}\mu \ge \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu.$$

**Remark:** In fact, Prop. 1 still holds (under the same proof as done in class) if we ask $f$ and $g$ to be "positive" functions instead of "positive and measurable" functions (why?).

Now, we will prove the following:

**Prop 2.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f$ and $g$ be two positive measurable functions. Then

$$\int_X f + g \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu.$$

*Proof:* It remains to show

$$\int_X f + g \,\mathrm{d}\mu \le \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu.$$

We just need to prove the inequality above for those $f$ and $g$ satisfying: for any simple functions $f_n, g_n$ and $s_n$ satisfying $0 \leq f_n \leq f$, $0 \leq g_n \leq g$ and $0 \leq s_n \leq f+g$, the integrations of simple functions $f_n$, $g_n$ and $s_n$ are all finite.

In fact, if not, assume the integration of simple functions $f_n$ or $g_n$ above is inifinite. Then it follows that $\int_X f \, \mathrm{d}\mu$ or $\int_X g \, \mathrm{d}\mu$ is $+\infty$. Then we definitely have

$$\int_X f + g \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu.$$

If the the integration of the simple function $s_n$ above is $+\infty$, we will show that the inequality is still true. Assume $s_n = \sum_{i=1}^{K} \lambda_i \chi_{E_i}$ where each $\lambda_i > 0$.

As the integration of $s_n$ is infinite, without loss of generality, we assume that $\mu(E_1) = \infty$. Then $f + g \geq \lambda_1 > 0$ on $E_1$. Let $f' = \min\{f, 1\}$ and $g' = 1 - f'$ (or equivalently, $g' = \max\{1 - f, 0\}$). It follows that $0 \leq f' \leq f$, $0 \leq g' \leq g$ and $f' + g' = 1$. According to problem 4 of HW no. 5, we have

$$\int_X f' \, \mathrm{d}\mu + \int_X g' \, \mathrm{d}\mu = \infty,$$

which then implies that

$$\int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu = \infty.$$

Thus the inequality

$$\int_X f + g \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu$$

still holds in this case.

Now, we can just assume (why?) that for all $F \in \{f, g, f+g\}$,

$$\int_X F \, \mathrm{d}\mu = \sup\left\{ \int_X s \, \mathrm{d}\mu \colon s \text{ is simple}, 0 \leq s \leq F \text{ and } s > 0 \text{ holds on a set of finite measure} \right\}.$$

Under this assumption ,we will show that

$$\int_X f + g \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu.$$

That is, for any simple function $s$ with $0 \leq s \leq f + g$ and $s > 0$ holds on a set of finite measure, we have

$$\int_X f + g \,\mathrm{d}\mu \leq \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu.$$

Use $E$ to denote $\{x \in X \colon s(x) > 0\}$. Then $\mu(E) < \infty$.

Let $h_1 = \min(f, s)$ and let $h_2 = s - h_1$ (or equivalently, $h_2 = \max(s - f, 0)$). One can then check that $0 \leq h_1 \leq f$. When $f(x) < s(x)$, $h_2(x) = s(x) - f(x) \leq g(x)$. When $f(x) \geq s(x)$, $h_2(x) = 0 \leq g(x)$. Then we can conclude that $0 \leq h_2 \leq g$. From the definition of initegrations, it follows that

$$\int_X h_1 \,\mathrm{d}\mu \leq \int_X f \,\mathrm{d}\mu \quad \text{and} \quad \int_X h_2 \,\mathrm{d}\mu \leq \int_X g \,\mathrm{d}\mu.$$

Note that $s$ is zero outside $E$. As $h_1 + h_2 = s$ and both $h1$ and $h_2$ are positive, it follows that $0 \leq h_i \leq s$ for $i = 1, 2$. As $s$ is bounded, so is $h_1$ and $h_2$. As $s$ is zero outside $E$, so is $h_1$ and $h_2$.

Recall that in case the measure space is of finite measure and two measurable positive functions are bounded, we have proved (in Quiz no. 2) that the integration of their sum equals the sum of the integration of the functions. Then we have

$$\begin{aligned}
\int_X s \,\mathrm{d}\mu &= \int_X \chi_E \cdot s \,\mathrm{d}\mu \\
&= \int_E s \,\mathrm{d}\mu \\
&= \int_E h_1 + h_2 \,\mathrm{d}\mu \\
&= \int_E h_1 \,\mathrm{d}\mu + \int_E h_2 \,\mathrm{d}\mu \qquad \text{[ because } \mu(E) < \infty \text{ and } |h_i| \text{ are bounded ]} \\
&= \int_X h_1 \,\mathrm{d}\mu + \int_X h_2 \,\mathrm{d}\mu \\
&\leq \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu,
\end{aligned}$$

which finishes the proof.

Q.E.D.

**Remark:** Based on this result, for any $f, g \in L^1(X, \mathcal{M}, \mu)$, we can show (already done in class) that

$$\int_X f + g \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu.$$